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New results concerning monotone operators and nonlinear semigroups

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Our purpose is to describe here some recent developments in three different directions.

In §I we discuss a property of the range $R(A+B)$ of the sum of two monotone operators. Surprisingly, it turns out that in "many" cases $R(A+B)$ is "almost" equal to $R(A)+R(B)$. A number of applications to nonlinear partial differential equations are given.

In §II we prove some estimates showing that $(I+tA)^{-1}$ and $S(t)$ have the same modulus of continuity at $t=0$ ($S(t)$ denotes the semigroup generated by $-A$). Next we present some consequences.

In §III we give a very general form of the convergence theorem of Trotter - Kato - Neveu type for nonlinear semigroups.

§I "$R(A+B) \simeq R(A)+R(B)$" and applications

Let H be a real Hilbert space and let A and B be maximal monotone operators such that $A+B$ is again maximal monotone.

We say that two subsets K_1 and K_2 of H are almost equal ($K_1 \simeq K_2$) if K_1 and K_2 have the same closure and the same interior. We prove here, under various assumptions, that

$R(A+B) \simeq R(A)+R(B)$; we discuss here only the simplest forms (for more elaborate results see [7]).

Theorem 1 Suppose A and B are subdifferentials of convex functions. Then $R(A+B) \simeq R(A)+R(B)$.

Proof First we prove that $\overline{R(A+B)} = \overline{R(A)+R(B)}$; it is sufficient to verify that $R(A)+R(B) \subset \overline{R(A+B)}$. Given $f \in R(A)+R(B)$, there exist $\xi \in D(A)$ and $\eta \in D(B)$ such that $f \in A\xi + B\eta$. The equation

(1) $$\varepsilon u_\varepsilon + Au_\varepsilon + Bu_\varepsilon \ni f$$

has a unique solution u_ε. The conclusion follows provided we show that $\varepsilon u_\varepsilon \to 0$ as $\varepsilon \to 0$. Let $x \in D(A) \cap D(B)$ be fixed. Since A and B are cyclically monotone (see [21]) we have

(2) $$(Au_\varepsilon, u_\varepsilon - x) + (Ax, x - \xi) + (A\xi, \xi - u_\varepsilon) \geqslant 0$$

(3) $$(Bu_\varepsilon, u_\varepsilon - x) + (Bx, x - \eta) + (B\eta, \eta - u_\varepsilon) \geqslant 0$$

and therefore by adding (2) and (3) we obtain

$$(f - \varepsilon u_\varepsilon, u_\varepsilon - x) + C - (f, u_\varepsilon) \geqslant 0,$$

where C is independent of ε. Hence

$$\varepsilon |u_\varepsilon|^2 - \varepsilon(u_\varepsilon, x) \leqslant C'$$

and therefore $\sqrt{\varepsilon}|u_\varepsilon|$ remains bounded as $\varepsilon \to 0$.

Next we prove that $\mathrm{Int}[R(A)+R(B)] = \mathrm{Int}[R(A+B)]$. It is sufficient to check that $\mathrm{Int}[R(A)+R(B)] \subset R(A+B)$. Let $f \in \mathrm{Int}[R(A)+R(B)]$, so that a ball $B(f, \rho)$ is contained in $R(A)+R(B)$. For every $h \in H$ with $|h| < \rho$, there exist ξ

and η (depending on h) such that $f+h \in A\xi + B\eta$. Going back to (2) and (3) and adding them we obtain now

$$(f - \varepsilon u_\varepsilon, u_\varepsilon - x) + C(h) - (f+h, u_\varepsilon) \geq 0$$

where $C(h)$ depends on h, but is independent of ε.
Hence $(h, u_\varepsilon) \leq C(h)$ for every $h \in H$ with $|h| < \rho$. It follows from the uniform boundedness principle that $\{u_\varepsilon\}$ remains bounded as $\varepsilon \to 0$. Passing to the limit in (1) we conclude by standard methods that $f \in R(A+B)$.

Theorem 2 We suppose now that only A is the subdifferential of a convex function, but $D(B) \subset D(A)$. Then $R(A+B) \simeq R(A) + R(B)$.

Proof We proceed as in the proof of Theorem 1.
First let $f \in R(A+B)$ i.e. $f \in A\xi + B\eta$; let u_ε be the solution of (1). We have

(4) $$(Au_\varepsilon, u_\varepsilon - \eta) + (A\eta, \eta - \xi) + (A\xi, \xi - u_\varepsilon) \geq 0$$

(5) $$(Bu_\varepsilon, u_\varepsilon - \eta) + (B\eta, \eta - u_\varepsilon) \geq 0.$$

By adding (4) and (5) we obtain

$$(f - \varepsilon u_\varepsilon, u_\varepsilon - \eta) + C - (f, u_\varepsilon) \geq 0$$

and hence

$$\varepsilon |u_\varepsilon|^2 - \varepsilon(u_\varepsilon, \eta) \leq C'.$$

Next suppose $f \in \mathrm{Int}[R(A) + R(B)]$; we obtain now, as in the proof of Theorem 1

$$(f - \varepsilon u_\varepsilon, u_\varepsilon - \eta) + C(h) - (f+h, u_\varepsilon) \geq 0$$

i.e. $(h, u_\varepsilon) \leq C'(h)$.

Theorem 3 Suppose A is a subdifferential of a convex function φ and let B be a maximal monotone operator such that

(6) $$\varphi((I+\lambda B)^{-1}x) \leq \varphi(x) \qquad \forall\lambda > 0, \ \forall x \in D(\varphi).$$

Then $R(A+B) \simeq R(A)+R(B)$.

Remark We know (see [4]) that (6) implies that $A+B$ is maximal monotone.

Proof Let $f \in R(A)+R(B)$ and let u_ε be the solution of (1). It follows easily from (6) that $\varepsilon|u_\varepsilon|$, $|Au_\varepsilon|$ and $|Bu_\varepsilon|$ remain bounded as $\varepsilon \to 0$. Next we have

(7) $$(Au_\varepsilon - A\xi, u_\varepsilon - \xi) \geqslant 0$$

(8) $$(Bu_\varepsilon - B\eta, u_\varepsilon - \eta) \geqslant 0.$$

Hence, by adding (7) and (8) we obtain

$$(f - \varepsilon u_\varepsilon, u_\varepsilon) - (f, u_\varepsilon) + C \geqslant 0$$

i.e. $\varepsilon|u_\varepsilon|^2 \leqslant C$. Suppose now that $f \in \mathrm{Int}[R(A)+R(B)]$, with the same argument as above we have

$$(f - \varepsilon u_\varepsilon, u_\varepsilon) - (f+h, u_\varepsilon) + C(h) \geqslant 0$$

i.e. $(h, u_\varepsilon) \leqslant C(h)$ for $|h| < \rho$.

Some applications

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a monotone nondecreasing continuous function such that $\beta(0) = 0$. Consider the equation (for a given $f \in L^2(\Omega)$):

(9) $$-\Delta u + \beta(u) = f \text{ on } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

Theorem 4 A necessary condition for the existence of a solution of (9) is that $\frac{1}{|\Omega|}\int_\Omega f(x)dx \in \overline{R(\beta)}$. A sufficient condition is that $\frac{1}{|\Omega|}\int_\Omega f(x)dx \in \operatorname{Int} R(\beta)$.

<u>Proof</u> The necessary condition is clear by integrating (9) on Ω. In order to prove the sufficient condition we apply Theorem 1 in $H = L^2(\Omega)$ with

$$A = -\Delta, \quad D(A) = \left\{u \in H^2(\Omega);\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}$$

$$B = \beta, \quad D(B) = \left\{u \in L^2(\Omega);\ \beta(u) \in L^2(\Omega)\right\}.$$

Both A and B are subdifferentials of convex functions; also $A+B$ is maximal monotone. It is well known that $R(A) = \left\{f \in L^2(\Omega);\ \int_\Omega f(x)dx = 0\right\}$. Finally if $\frac{1}{|\Omega|}\int_\Omega f(x)dx \in \operatorname{Int} R(\beta)$, then $f \in \operatorname{Int}[R(A)+R(B)]$. Indeed for $g \in L^2(\Omega)$ we have

$$g = \left(g - \frac{1}{|\Omega|}\int_\Omega g(x)dx\right) + \frac{1}{|\Omega|}\int_\Omega g(x)dx .$$

And so it is clear that $g \in R(A)+R(B)$ as soon as

$\left|\frac{1}{|\Omega|}\int_\Omega g(x)dx - \frac{1}{|\Omega|}\int_\Omega f(x)dx\right| \leq |\Omega|^{-\frac{1}{2}}\|f-g\|_{L^2}$ is small enough.

<u>Remark</u> Theorem 4 is related to a number of results of Schatzman [22], Hess [13], Landesman - Lazer [17], Nirenberg [19] etc... The method used in the proofs of Theorems 1 - 3 can be easily extended to include most results known about "semi coercive" problems.

Let $\mathcal{H}$ be a Hilbert space and let φ be a convex function on $\mathcal{H}$. Given $f \in L^2(0, T; \mathcal{H})$ consider the equation

(10) $\qquad \frac{du}{dt} + \partial\varphi(u) \ni f \quad \text{on} \quad (0, T), \qquad u(0) = u(T)$.

Theorem 5 A necessary condition for the existence of a solution of (10) is that $\frac{1}{T}\int_0^T f(t)dt \in \overline{R(\partial\varphi)}$. A sufficient condition is that $\frac{1}{T}\int_0^T f(t)dt \in \operatorname{Int} R(\partial\varphi)$.

Proof Since $\overline{R(\partial\varphi)}$ is convex, the necessary condition follows from the integration of (10). For the sufficient condition we apply Theorem 3 in $H = L^2(0, T; \mathcal{H})$ with $A = \partial\varphi$ i.e. $f \in Au$ provided $f, u \in H$ and $f(t) \in \partial\varphi(u(t))$ a.e. and with $B = \frac{d}{dt}$, $D(B) = \left\{u \in H, \ \frac{du}{dt} \in H \text{ and } u(0) = u(T)\right\}$. It is well known that A is a subdifferential of a convex function in H, that B is maximal monotone and that (6) holds. The assumption $\frac{1}{T}\int_0^T f(t)dt \in \operatorname{Int} R(\partial\varphi)$ implies that $f \in \operatorname{Int}[R(A)+R(B)]$. Indeed, note that $R(B) = \left\{f \in H; \int_0^T f(t)dt = 0\right\}$. For $g \in H$ we can write

$$g = \left(g - \frac{1}{T}\int_0^T g(t)dt\right) + \frac{1}{T}\int_0^T g(t)dt \in R(A) + R(B)$$

provided $\|g - f\|_H$ is small enough.

Theorem 6 Let H be a Hilbert space and let K be a maximal monotone operator in H with $D(K) = H$. Let F be the subdifferential of a convex function on H with $D(F) = H$. Then $R(I + KF) = H$.

Proof Given $f \in H$ we want to solve $u + KFu = f$ i.e.

$-K^{-1}(f-u) + Fu \ni 0$. We apply Theorem 2 with $A = F$ and $Bu = -K^{-1}(f-u)$ so that B is maximal monotone; it follows that $R(A+B) \simeq R(A) + R(B)$. However $R(B) = -D(K) = H$ and therefore $R(A+B) = H$.

Remark Results related to Theorem 6 were obtained in [6].

§ II.1 Comparative behavior of $(I+tA)^{-1}$ and $S(t)$ near $t = 0$

1. The Hilbert space case

Suppose H is a Hilbert space and let A be a maximal monotone operator; let $S(t)$ be the semigroup generated by $-A$ in the sense of Kato - Komura (see e.g. [23] or [4]).
For $x \in \overline{D(A)}$ and $y \in D(A)$ we have

$$|x - S(t)x| \leq 2|x-y| + |y - S(t)y| \leq 2|x-y| + t|A^{\circ}y| .$$

Choosing $y = J_\lambda x = (I + \lambda A)^{-1}x$ we get

$$|x - S(t)x| \leq \left(2 + \frac{t}{\lambda}\right)|x - J_\lambda x| \tag{11}$$

and in particular, for $\lambda = t$, we obtain

$$|x - S(t)x| \leq 3|x - J_t x| . \tag{12}$$

In case $A = \partial\varphi$ we can show (see [5]) that

$$|x - J_t x| \leq \left(1 + \frac{1}{\sqrt{2}}\right)|x - S(t)x| \tag{13}$$

(the best constants are not known).
For general monotone operators an inequality of the kind (13) does not hold (consider for example in $H = \mathbb{R}^2$, $A =$ a rotation

by $\pi/2$). However one can obtain a "substitute" for (13) in the general case as follows:

Theorem 7 Let A be a general maximal monotone operator; then we have

(14) $$|x - J_t x| \leqslant \frac{2}{t}\int_0^t |x - S(\tau)x|\, d\tau\,, \quad \forall x \in \overline{D(A)}, \quad \forall t > 0.$$

Remark It is clear that the constant 2 in (14) can not be improved. Otherwise we would have for $x \in D(A)$, $|x - J_t x| \leqslant \frac{C}{t}\int_0^t \tau |A^\circ x| d\tau = \frac{C}{2}|A^\circ x| t$ and as $t \to 0$, $|A^\circ x| \leqslant \frac{C}{2}|A^\circ x|$ with $C < 2$.

Proof Clearly, it is sufficient to prove (14) for $x \in D(A)$. Let $u(t) = S(t)x$; by the monotonicity of A, we have for $v \in D(A)$

(15) $$\left(Av + \frac{du}{dt}(t),\ v - u(t)\right) \geqslant 0\ .$$

Integrating (15) on $(0, t)$ we obtain

(16) $$\frac{1}{2}|u(t) - v|^2 - \frac{1}{2}|x - v|^2 \leq \int_0^t (Av,\ v - u(\tau))\, d\tau =$$

$$= t(Av,\ v - x) + \int_0^t (Av,\ x - u(\tau))\, d\tau\ .$$

Thus $\left|\frac{1}{2}u(t) - v\right|^2 - \frac{1}{2}|x - v|^2 \leqslant t(Av,\ v - x) + |Av|\int_0^t |x - u(\tau)|\, d\tau.$

Choosing $v = J_t x$ we get

$$\frac{1}{2}|u(t) - J_t x|^2 - \frac{1}{2}|x - J_t x|^2 \leqslant -|x - J_t x|^2 + \frac{|x - J_t x|}{t}\int_0^t |x - u(\tau)|\, d\tau\ ,$$

and (14) follows.

<u>Remark</u> Combining (12) and (14) we see that $|x - J_t x|$ and $|x - S(t)x|$ have the same modulus of continuity at $t = 0$. Also, using Hardy's inequality we can deduce that for $1 \geq \alpha > 0$ and $1 \leq p \leq \infty$

$$\left\| \frac{x - S(t)x}{t^\alpha} \right\|_{L^p_*} \leq 3 \left\| \frac{x - J_t x}{t^\alpha} \right\|_{L^p_*} \quad \text{and}$$

$$\left\| \frac{x - J_t x}{t^\alpha} \right\|_{L^p_*} \leq \frac{2}{1+\alpha} \left\| \frac{x - S(t)x}{t^\alpha} \right\|_{L^p_*}$$

where $L^p_* = L^p([0, 1], H; \frac{dt}{t})$. These inequalities are useful in the study of nonlinear interpolation classes (see [3]).

In a "similar spirit" we have the following

<u>Theorem 8</u> Let A be a general maximal monotone operator. For $x \in \overline{D(A)}$, $\lambda > 0$ and $t > 0$ we set

$$y_{\lambda,t} = (I + \frac{\lambda}{t}(I - S(t)))^{-1} x .$$

Then

$$|y_{\lambda,t} - J_\lambda x|^2 \leq |x - J_\lambda x| \, \frac{2}{t} \int_0^t |x - S(\tau)x| \, d\tau . \tag{17}$$

<u>Remark</u> Let $\omega(t) = \operatorname*{Sup}_{0 \leq \tau \leq t} |x - S(\tau)x|$. By a result of Kato [14] (see also [4] Lemma 4.2) we know that for every integer n

$$|y_{\lambda,t} - y_{\lambda,t/n}|^2 \leq 2\,\omega(t)\,|y_{\lambda,t/n} - x| .$$

Using the fact that $y_{\lambda,s} \longrightarrow J_\lambda x$ as $s \longrightarrow 0$ (see e.g. [4] Proposition 4.1) we obtain as $n \longrightarrow \infty$

$$|y_{\lambda,t} - J_\lambda x|^2 \leq 2\,\omega(t)\,|J_\lambda x - x| . \tag{18}$$

Such an inequality follows also directly from (17).

<u>Proof</u> We apply (16) with x replaced by $y_{\lambda,t}$ and v by $J_\lambda x$. Thus

$$(19)\quad \frac{1}{2}|S(t)y_{\lambda,t} - J_\lambda x|^2 - \frac{1}{2}|y_{\lambda,t} - J_\lambda x|^2 \leqslant \int_0^t \left(\frac{x - J_\lambda x}{\lambda}, J_\lambda x - S(\tau)y_{\lambda,t}\right) d\tau .$$

However $S(t)y_{\lambda,t} = (1+\frac{t}{\lambda})y_{\lambda,t} - \frac{t}{\lambda}x$ and so

$$(20)\quad |S(t)y_{\lambda,t} - J_\lambda x|^2 \geqslant |y_{\lambda,t} - J_\lambda x|^2 + \frac{2t}{\lambda}(y_{\lambda,t} - J_\lambda x, y_{\lambda,t} - x).$$

On the other hand

$$(21)\quad (x - J_\lambda x, J_\lambda x - S(\tau)y_{\lambda,t}) = -|x - J_\lambda x|^2 + (x - J_\lambda x, x - S(\tau)y_{\lambda,t})$$
$$\leqslant -|x - J_\lambda x|^2 + |x - J_\lambda x|(|x - S(\tau)x| + |x - y_{\lambda,t}|).$$

We deduce from (19), (20) and (21) that

$$\frac{t}{\lambda}(y_{\lambda,t}-J_\lambda x, y_{\lambda,t}-x) \leqslant -\frac{t}{\lambda}|x-J_\lambda x|^2 + \frac{t}{\lambda}|x-J_\lambda x|\,|x-y_{\lambda,t}| + \frac{|x-J_\lambda x|}{\lambda}\int_0^t |x-S(\tau)x|\, d\tau .$$

Therefore

$$|x-J_\lambda x|^2 + (y_{\lambda,t}-J_\lambda x, y_{\lambda,t}-x) \leqslant |x-J_\lambda x|\,|x-y_{\lambda,t}| + |x-J_\lambda x|\frac{1}{t}\int_0^t |x-S(\tau)x|\, d\tau$$

i.e. $|a|^2 + (b-a, b) \leqslant |a|\,|b| + |x - J_\lambda x|\frac{1}{t}\int_0^t |x - S(\tau)x|\, d\tau$

with $a = x - J_\lambda x$ and $b = x - y_{\lambda,t}$. Hence

$$\frac{1}{2}|a-b|^2 = \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2 - (a,b) \leqslant$$
$$-\frac{1}{2}|a|^2 - \frac{1}{2}|b|^2 + |a|\,|b| + |x-J_\lambda x|\,\frac{1}{t}\int_0^t |x-S(\tau)x|\, d\tau$$

and $\frac{1}{2}|a-b|^2 \leqslant |x-J_\lambda x|\frac{1}{t}\int_0^t |x-S(\tau)x|\, d\tau .$

II.2 The Banach space case

Let X be a general Banach space and let A be an m-accretive operator on X. Let $S(t)$ be the semigroup generated by $-A$ in the sense of Crandall - Liggett (see [10] or [23]). Clearly we have as in § II.1

$$(22) \qquad \|x - S(t)x\| \leq (2 + \frac{t}{\lambda}) \|x - J_\lambda x\| .$$

We don't know whether the exact analogue of (14) holds true. However we can prove the following

<u>Theorem 9</u> For every $x \in \overline{D(A)}$, $t > 0$ and $\lambda > 0$ we have

$$(23) \qquad \|x - J_\lambda x\| \leq (1 + \frac{\lambda}{t}) \frac{2}{t} \int_0^t \|x - S(\tau)x\| \, d\tau$$

and in particular

$$(24) \qquad \|x - J_t x\| \leq \frac{4}{t} \int_0^t \|x - S(\tau)x\| d\tau .$$

<u>Proof</u> As usual we denote for $x, y \in X$

$$\tau(x, y) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda}(\|x+\lambda y\| - \|x\|) = \inf_{\lambda > 0} \frac{1}{\lambda}(\|x+\lambda y\| - \|x\|) .$$

The analogue of (16) becomes now (see [10] or [2] for equivalent forms):

$$(25) \qquad \|S(t)x - v\| - \|v - x\| \leq \int_0^t \tau(v - S(s)x, Av) ds$$

for every $v \in D(A)$.

However we have for every $\lambda > 0$

$$(26) \qquad \tau(v - S(s)x,\ Av) \leq \frac{1}{\lambda}(\|v - S(s)x + \lambda Av\| - \|v - S(s)x\|) .$$

If we choose in (26) $v = J_\lambda x$ we obtain

(27) $\tau(J_\lambda x - S(s)x,\ A_\lambda x) \leqslant \frac{1}{\lambda}(\|x-S(s)x\| - \|J_\lambda x-S(s)x\|)$

and by (25) we get

(28) $\|S(t)x - J_\lambda x\| - \|J_\lambda x - x\| \leqslant \frac{1}{\lambda}\int_0^t (\|x-S(s)x\| - \|J_\lambda x-S(s)x\|)ds.$

But $-\|J_\lambda x-S(s)x\| \leqslant \|x-S(s)x\| - \|x-J_\lambda x\|$ and therefore (28) leads to

$$-\|x-S(s)x\| \leqslant \frac{1}{\lambda}\int_0^t \|x-S(s)x\|ds + \frac{1}{\lambda}\int_0^t (\|x-S(s)x\|ds - \frac{t}{\lambda}\|x-J_\lambda x\|$$

i.e.

(29) $\|x - J_\lambda x\| \leqslant \frac{\lambda}{t}\|x-S(t)x\| + \frac{2}{t}\int_0^t \|x-S(s)x\|ds$.

Finally note that

(30) $\|x - S(t)x\| \leqslant \frac{2}{t}\int_0^t \|x - S(s)x\| ds$;

indeed

$$\|S(t)x - \frac{1}{t}\int_0^t S(s)x\,ds\| \leqslant \frac{1}{t}\int_0^t \|S(t)x - S(s)x\|ds$$

$$\leqslant \frac{1}{t}\int_0^t \|S(t-s)x - x\|ds = \frac{1}{t}\int_0^t \|S(s)x - x\|ds ,$$

and so

$$\|x-S(t)x\| \leqslant \|x - \frac{1}{t}\int_0^t S(s)x\,ds\| + \frac{1}{t}\int_0^t \|S(s)x-x\|ds \leqslant \frac{2}{t}\int_0^t \|x-S(s)x\|ds.$$

Combining (29) and (30) we obtain (23).

Remarks:

1) I would like to thank Prof. M. Crandall, Y. Konishi and I. Miyadera for stimulating discussions concerning Theorem 9. After our first result was obtained ($\|x-J_t x\| \leqslant \frac{2}{t}\int_0^{2t} \|x-S(\tau)x\|d\tau$), I. Miyadera showed that $\|x - J_t x\| \leqslant \frac{6}{t}\int_0^t \|x - S(\tau)x\| d\tau$ and

Y. Konishi got $\|x - J_t x\| \leq \frac{4}{t}\int_0^t \|x - S(\tau)x\| d\tau$.

2) Using (22) and (23) one can prove directly the following result of M. Crandall [9]:

$$\limsup_{t\downarrow 0} \frac{\|x - S(t)x\|}{t} = \lim_{\lambda\downarrow 0} \frac{\|x - J_\lambda x\|}{\lambda}.$$

Indeed let $\alpha = \limsup_{t\downarrow 0} \frac{\|x - S(t)x\|}{t}$; and so $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $0 < t < \delta$

$$\|x - S(t)\| \leq t(\alpha + \varepsilon).$$

From (23) we have for $0 < t < \delta$ and every $\lambda > 0$

$$\|x - J_\lambda x\| \leq (1 + \frac{\lambda}{t})\frac{2}{t}(\alpha + \varepsilon)\int_0^t \tau d\tau = (\lambda + t)(\alpha + \varepsilon).$$

It follows that $\|x - J_\lambda x\| \leq \lambda(\alpha + \varepsilon)$ for every $\lambda > 0$ and $\varepsilon > 0$. Next let $\beta = \lim_{\lambda\downarrow 0} \frac{\|x - J_\lambda x\|}{\lambda}$; and so $\forall \varepsilon > 0$ $\exists \delta > 0$ such that for $0 < \lambda < \delta$

$$\|x - J_\lambda x\| \leq \lambda(\beta + \varepsilon).$$

From (22) we get for $0 < \lambda < \delta$ and every $t > 0$

$$\|x - S(t)x\| \leq (2 + \frac{t}{\lambda})\lambda(\beta + \varepsilon) = (t + 2\lambda)(\beta + \varepsilon).$$

Hence $\|x - S(t)x\| \leq t\beta$ for every $t > 0$.

3) In general for $x \in \overline{D(A)}$, $\frac{|x - S(t)x|}{|x - J_t x|}$ does not necessarily converge to 1 as $t \to 0$.

Consider for example in $H = \mathbb{R}$, $Au = \frac{-1}{u}$ for $u > 0$ and $Au = \phi$ for $u \leq 0$. In this case $J_t 0 = \sqrt{t}$ and $S_t 0 = \sqrt{2t}$ (slightly more complicated examples were built previously by A. Plant and L. Veron).

4) In view of the example built by Crandall - Liggett in [11]

one can not expect to extend Theorem 8 to Banach spaces (or even to $\mathbb{R}^3$ with some Banach norm) since $y_{\lambda,t}$ does not necessarily converge to a limit as $t \to 0$.

II.3 An application to the characterization of compact semi-groups.

Let A be an m-accretive operator in a general Banach space X and let $S(t)$ be the semigroup generated by $-A$.

Theorem 10. The following properties are equivalent.

(31) For every $t > 0$, $S(t)$ is compact i.e. $S(t)$ maps bounded sets of $\overline{D(A)}$ into compact sets of X

(32) $\begin{cases} \text{(32a) For every } \lambda > 0,\ (I + \lambda A)^{-1} \text{ is compact i.e.} \\ \text{maps bounded sets of } X \text{ into compact sets of } X \\ \text{(32b) For every bounded set } B \text{ in } \overline{D(A)} \text{ and every } t_0 > 0 \\ \text{the mappings } t \mapsto S(t)x \text{ are equicontinuous at } t = t_0 \\ \text{as } x \in B. \end{cases}$

Remarks

1) Theorem 10 is due to A. Pazy [20] in the linear case and to Y. Konishi [15] in the nonlinear Hilbert case (his proof relies on a consequence of (18) and could not be extended to Banach spaces)

2) It is obvious that (32a) is equivalent to

(32a') $(I + A)^{-1}$ is compact

and also to

(32a") For every $M > 0$ the set

$$\{x \in D(A);\ \|x\| \leq M \text{ and } \|y\| \leq M \text{ for some } y \in Ax\}$$

is relatively compact in X.

Proof (31) $\Longrightarrow$ (32a)

Let λ be fixed and let $x \in X$; we have for every $t \geq 0$

$$\|J_\lambda x - S(t)J_\lambda x\| \leq t\|A_\lambda x\| = \frac{t}{\lambda}\|x - J_\lambda x\|.$$

Let B be a bounded set in X; given $\varepsilon > 0$, choose t_0 so small that

$$\frac{t_0}{\lambda}\|x - J_\lambda x\| < \varepsilon/2 \quad \text{for } x \in B.$$

Since $J_\lambda(B)$ is bounded in $\overline{D(A)}$, it follows from (31) that $S(t_0)J_\lambda(B)$ is relatively compact. Thus $S(t_0)J_\lambda(B)$ can be covered by a finite union $\bigcup_i B(x_i, \varepsilon/2)$. Hence $J_\lambda(B) \subset \bigcup_i B(x_i, \varepsilon)$ and consequently $J_\lambda(B)$ is precompact.

(31) $\Longrightarrow$ (32b)

Using (31) we have only to prove that the mappings $t \longmapsto S(t)x$ are equicontinuous at $t = \frac{t_0}{2}$ as $x \in K$, K compact ($K = \overline{S(\frac{t_0}{2})B}$). This follows directly from the fact that for each fixed x, $t \mapsto S(t)x$ is continuous and that $x \mapsto S(t)x$ is a contraction.

(32a) + (32b) $\Longrightarrow$ (31)

Fix a $t_0 > 0$ and let B be a bounded set in $\overline{D(A)}$. By (32b), for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|S(t)x - S(t_0)x\| < \varepsilon \quad \text{for } |t - t_0| \leq \delta \text{ and } x \in B.$$

We deduce from (23) that for $x \in B$ and $\lambda > 0$,

$$\|S(t_0)x - J_\lambda S(t_0)x\| \leq (1 + \frac{\lambda}{t})\frac{2}{t}\int_0^t \|S(t_0)x - S(\tau + t_0)x\|\, d\tau$$

$$\leq (1+\frac{\lambda}{t})\, 2\varepsilon \qquad \text{for every } \; 0 < t \leq \delta \, .$$

In particular for $0 < \lambda \leq \delta$ and $x \in B$ we have

$$\| S(t_0)x - J_\lambda S(t_0)x \| \leq 4\varepsilon \, .$$

Since $J_\delta S(t_0)B$ is relatively compact it can be covered by a finite union $\bigcup_i B(x_i, \varepsilon)$. Hence $S(t_0)B$ can also be covered by a finite union of balls of radius 5ε and thus $S(t_0)B$ is precompact.

<u>Remark</u> Suppose H is a Hilbert space, φ is a convex function on H and let $A = \partial\varphi$. In this case (31) is equivalent to (32a) since (32b) is satisfied automatically. Indeed we have

$$|S(t)x - S(t_0)x| = |S(t - \frac{t_0}{2})y - S(\frac{t_0}{2})y| \leq |t - t_0|\, |A^\circ y|$$

where $y = S(\frac{t_0}{2})x$. On the other hand (see e.g. [4] Théorème 3.2) we know that

$$|A^\circ S(\frac{t_0}{2})x| \leq |A^\circ v| + \frac{2}{t_0}|x - v| \quad \text{for every } \; v \in D(A).$$

Therefore the mappings $t \mapsto S(t)x$ are equicontinuous at $t = t_0$ as x remains bounded.

In this case property (32a) is also equivalent to

(32a''') For every M the set

$$\{ x \in D(\varphi);\; |x| \leq M \text{ and } \varphi(x) \leq M \}$$

is relatively compact in H.

Indeed (32a''') $\Longrightarrow$ (32a''):

Let $E = \{x \in D(A);\; |x| \leq M \text{ and } |A^\circ x| \leq M\}$; for a fixed $v_0 \in D(\varphi)$ we have

$$\varphi(v_0) - \varphi(x) \geq (A^\circ x, v_0 - x)$$

and so $\varphi(x) \leq \varphi(v_0) + M(|v_0| + M) = M'$ when $x \in E$.

Conversely (32a) $\Longrightarrow$ (32a'''):

Let

$$F = \{x \in D(\varphi);\ |x| \leq M \text{ and } \varphi(x) \leq M\};$$

for $x \in F$ we have

$$\varphi(x) - \varphi(J_\lambda x) \geq (A_\lambda x, x - J_\lambda x) = \frac{1}{\lambda}|x - J_\lambda x|^2.$$

Therefore, since φ is bounded below by some affine function, we get for $x \in F$,

$$\frac{1}{\lambda}|x - J_\lambda x|^2 \leq M + C_1|J_\lambda x| + C_2 \leq M + C_1|x - J_\lambda x| + C_1 M + C_2.$$

Thus $|x - J_\lambda x| \leq \sqrt{\lambda(C_3\lambda + C_4)}$ for $x \in F$.

Given $\varepsilon > 0$ we choose $\lambda_0 > 0$ so small that $\sqrt{\lambda_0(C_3\lambda_0 + C_4)} < \varepsilon$. Since $J_{\lambda_0}(F)$ is relatively compact, it can be covered by a finite union $\bigcup_i B(x_i, \varepsilon)$ and then $F \subset \bigcup_i B(x_i, 2\varepsilon)$.

§ III. A convergence theorem for nonlinear semigroups

Let H be a Hilbert space; let $\{A_n\}_{n\geq 1}$ and A be maximal monotone operators. Let $\{S_n(t)\}_{n\geq 1}$ and $S(t)$ be the corresponding semigroups.

Our next result is a nonlinear version of the Theorem of Trotter - Kato - Neveu. A number of related results have been obtained previously by Miyadera - Oharu [18], Brezis - Pazy [8], Benilan [1], Goldstein [12], Kurtz [16] etc...

Theorem 11. The following properties are equivalent.

(33) $\forall x \in \overline{D(A)},\ \forall \lambda > 0 \quad (I + \lambda A_n)^{-1}x \longrightarrow (I + \lambda A)^{-1}x$

(34) $\forall x \in D(A) \quad \exists x_n \in D(A_n)$ such that $x_n \to x$ and

$A_n^\circ x_n \to A^\circ x$

(35) $\forall x \in \overline{D(A)} \quad \exists x_n \in \overline{D(A_n)}$ such that $x_n \to x$ and $\forall t \geqslant 0$

$S_n(t)x_n \to S(t)x$.

In addition the convergence in (33) (resp. (35)) is uniform for bounded λ (resp. bounded t).

The proof of Theorem 11 is divided into four parts

Part A (33) $\Longrightarrow$ (34)

Part B (34) $\Longrightarrow$ (33)

Part C (33) $\Longrightarrow$ (35)

Part D (35) $\Longrightarrow$ (33).

<u>Part A</u> (33) $\Longrightarrow$ (34)

Let $x \in D(A)$; given $\varepsilon > 0$ there is a $\lambda > 0$ such that

$$|x - (I + \lambda A)^{-1}x| < \varepsilon/2$$

$$|A^\circ x - A_\lambda x| < \varepsilon/2 .$$

Next, by (33) there is an integer N such that for $n \geqslant N$

$$|(I + \lambda A_n)^{-1}x - (I + \lambda A)^{-1}x| < \varepsilon/2$$

$$|(A_n)_\lambda x - A_\lambda x| < \varepsilon/2 .$$

Combining these estimates we see that given $\varepsilon > 0$ there is an integer $N(\varepsilon)$ and sequences $u_n(\varepsilon) = (I + \lambda A_n)^{-1}x$ and $f_n(\varepsilon) = (A_n)_\lambda x$ such that $[u_n(\varepsilon), f_n(\varepsilon)] \in G(A_n)$ and for $n \geqslant N(\varepsilon)$, $|u_n(\varepsilon) - x| < \varepsilon$, $|f_n(\varepsilon) - A^\circ x| < \varepsilon$. Let $N_k = N\left(\frac{1}{k}\right)$; we can always assume that N_k is increasing to ∞ .

We define the sequences x_n and g_n by $x_n = u_n(\frac{1}{k})$ and $g_n = f_n(\frac{1}{k})$ for $N_k \leq n < N_{k+1}$. Therefore $[x_n, g_n] \in G(A_n)$ and for $N_k \leq n < N_{k+1}$ we have $|x_n - x| < \frac{1}{k}$ and $|g_n - A^\circ x| < \frac{1}{k}$. Consequently $x_n \to x$ and $g_n \to A^\circ x$; we are going to prove now that $A_n^\circ x_n \to A^\circ x$. Indeed $|A_n^\circ x_n| \leq |g_n|$ and thus for a subsequence we get $A_{n_j}^\circ x_{n_j} \rightharpoonup h$. Let $v \in D(A)$; by the monotonicity of A_n we have

$$((A_n)_\lambda v - A_n^\circ x_n, \ (1 + \lambda A_n)^{-1} v - x_n) \geq 0 .$$

At the limit as $n_j \to \infty$ we obtain

$$(A_\lambda v - h, \ (I + \lambda A)^{-1} v - x) \geq 0 .$$

Next we pass to the limit as $\lambda \to 0$:

$$(A^\circ v - h, \ v - x) \geq 0 \qquad \forall \, v \in D(A) .$$

Therefore $h \in Ax$ (see e.g. [4] Proposition 2.7). Since on the other hand $|h| \leq |A^\circ x|$ we have $h = A^\circ x$. By the uniqueness of the limit, and the fact that $\limsup |A_n^\circ x_n| \leq |A^\circ x|$ we conclude that $A_n^\circ x_n \to A^\circ x$.

Part B (34) $\Rightarrow$ (33)

Without loss of generality we may assume that $\lambda = 1$. Let $x \in \overline{D(A)}$ and let $u_n = (I + A_n)^{-1} x$. Given $y \in D(A)$, let $y_n \in D(A_n)$ be the sequence given by (34) so that $y_n = (I + A_n)^{-1}(y_n + A_n^\circ y_n)$. Therefore $|u_n - y_n| \leq |x - y_n - A_n^\circ y_n|$ and thus u_n is bounded. For a subsequence $u_{n_j} \rightharpoonup u$; by the monotonicity of A_n we have

$$(x - u_n - A_n^\circ y_n, \ u_n - y_n) \geq 0 . \tag{36}$$

Passing to the limit in (36) we obtain

(37) $$(x - u - A^\circ y,\ u - y) \geqslant 0 \quad \forall y \in D(A).$$

In (37) we choose $y = (I + \lambda A)^{-1}u$ and so

$$(x - u,\ u - J_\lambda u) \geqslant \lambda (A^\circ J_\lambda u,\ A_\lambda u) \geqslant 0 .$$

As $\lambda \to 0$ we see that

$$(x - u,\ u - \mathrm{Proj}_{\overline{D(A)}}\, u) \geqslant 0 .$$

On the other hand since $x \in \overline{D(A)}$ we have

$$(\mathrm{Proj}_{\overline{D(A)}}\, u - x,\ u - \mathrm{Proj}_{\overline{D(A)}}\, u) \geqslant 0$$

and consequently $u = \mathrm{Proj}_{\overline{D(A)}}\, u$ i.e. $u \in \overline{D(A)}$. Going back to (37) we deduce now from [4] Proposition 2.7 that $x - u \in Au$ i.e. $u = (I + A)^{-1}x$. By the uniqueness of the limit we have in fact $u_n \rightharpoonup (I + A)^{-1}x$.

It follows from (36) that for every $y \in D(A)$

$$\limsup |u_n|^2 \leqslant (x, u-y) + (u, y) + (A^\circ y, y-u).$$

In particular if we take $y = u$ we get

$$\limsup |u_n|^2 \leqslant |u|^2 \quad \text{and thus} \quad u_n \to u .$$

The convergence in (33) is uniform in λ as λ remains bounded:

Without loss of generality we may assume that $x \in D(A)$ and let $x_n \in D(A_n)$ with $x_n \to x$ and $A_n^\circ x_n \to A^\circ x$. We have

$$|(I + \lambda A_n)^{-1}x_n - (I + \mu A_n)^{-1}x_n| \leqslant |\lambda - \mu|\, |A_n^\circ x_n| .$$

Therefore the functions $f_n(\lambda) = (I + \lambda A_n)^{-1}x_n$ are uniformly lipschitz continuous on $[0, +\infty)$. Since they converge simply to $(I + \lambda A)^{-1}x$ as $n \to +\infty$, we conclude that the convergence is uniform in λ as λ remains in a bounded interval.

<u>Part C</u> (33) $\Longrightarrow$ (35)

Without loss of generality we may assume that $x \in D(A)$. By (34) we have a sequence $x_n \in D(A_n)$ such that $x_n \to x$ and $A_n^\circ x_n \to A^\circ x$. We are going to prove that $S_n(t)x_n \to S(t)x$. It is known (see e.g. [4] Corollaire 4.4) that

$$\left| S_n(t)x_n - (I + \frac{t}{k}A_n)^{-k} x_n \right| \leq \frac{2t}{\sqrt{k}} \left| A_n^\circ x_n \right| \leq \frac{2tM}{\sqrt{k}}$$

and

$$\left| S(t)x - (I + \frac{t}{k}A)^{-k} x \right| \leq \frac{2t}{\sqrt{k}} \left| A^\circ x \right| \leq \frac{2tM}{\sqrt{k}}$$

where $M = \operatorname{Sup}_n \left| A_n^\circ x_n \right|$. Given $\varepsilon > 0$, we first fix k large enough so that $\frac{2Mt}{\sqrt{k}} < \varepsilon$. Next observe, by induction, that for every integer N and for every sequence $u_n \to u$ with $u \in \overline{D(A)}$ then $(I + \lambda A_n)^{-N} u_n \longrightarrow (I + \lambda A_n)^{-N} u$, as $n \to +\infty$. Thus

$$\left| S_n(t)x_n - S(t)x \right| \leq 2\varepsilon + \left| (I + \frac{t}{k}A_n)^{-k} x_n - (I + \frac{t}{k}A)^{-k} x \right| \leq 3\varepsilon$$

provided n is large enough.

Finally (35) holds true uniformly in t as t remains bounded since (33) holds true uniformly in λ as λ remains bounded.

<u>Part D</u> (35) $\Longrightarrow$ (33)

The proof relies on the following

<u>Lemma 1</u> Suppose (35) holds. Let $f_n \in \overline{D(A_n)}$ be such that $f_n \to f$ and $f \in \overline{D(A)}$. Then $\forall \lambda > 0$, $\forall t > 0$

$$u_n = (I + \frac{\lambda}{t}(I - S_n(t)))^{-1} f_n \longrightarrow u = (I + \frac{\lambda}{t}(I - S(t)))^{-1} f.$$

<u>Proof of Lemma 1</u> By (35) there exists a sequence $x_n \in \overline{D(A_n)}$ such that $x_n \to u$ and $S_n(t)x_n \longrightarrow S(t)u$. Writing the monotonicity of $I - S_n(t)$ we have

$$((u_n - S_n(t)u_n) - (x_n - S_n(t)x_n),\ u_n - x_n) \geq 0$$

and therefore

$$\left(\frac{u - u_n}{\lambda} + \delta_n , \; u_n - x_n\right) \geqslant 0$$

where $\quad \delta_n = \dfrac{f_n - f}{\lambda} + \dfrac{u - x_n}{t} + \dfrac{S_n(t)x_n - S(t)u}{t} \quad$ and $\quad \delta_n \longrightarrow 0$.

Hence

$$\frac{1}{\lambda}|u_n - u|^2 \leqslant |\delta_n|\,|u_n - u| + |\delta_n|\,|u - x_n| + \frac{1}{\lambda}|u - u_n|\,|u - x_n| \;,$$

and consequently $u_n \rightarrow u$ as $n \longrightarrow \infty$.

Lemma 2. Let $x_n \in \overline{D(A_n)}$ be a sequence such that $x_n \rightarrow x$ with $x \in \overline{D(A)}$ and $S_n(t)x_n \rightarrow S(t)x$ for every $t \geqslant 0$. Then for every T there exists a constant K such that $|(I + \lambda A_n)^{-1} x_n| \leqslant K$ and $|S_n(t)x_n| \leqslant K$ for every $0 < \lambda < T$, for every $0 < t < T$ and every n.

Proof of Lemma 2 Let $M = \sup\limits_{0 \leqslant t \leqslant 1} |S(t)x|$ and let

$$E_n = \left\{ t \in [0, 1] ; \; |S_p(t)x_p| \leqslant M + 1 \;\text{ for every }\; p \geqslant n \right\}.$$

Clearly E_n is closed and $\bigcup\limits_{n=1}^{\infty} E_n = [0, 1]$; it follows from Baire's theorem that $\operatorname{Int} E_N \neq \phi$ for some N. Let $[t_0, t_0 + h] \subset E_N$ so that

$$|S_p(t)x_p| \leqslant M + 1 \quad \text{for } n \geqslant N \text{ and } t_0 \leqslant t \leqslant t_0 + 1.$$

It follows from Theorem 9 that

$$|S_n(t_0)x_n - (I + \lambda A_n)^{-1} S_n(t_0)x_n| \leqslant (1 + \frac{\lambda}{h}) \frac{2}{h} \int_0^h |S_n(t_0)x_n - S_n(t_0 + \tau)x_n| d\tau .$$

Choosing $n \geqslant N$ we get

$$|(I + \lambda A_n)^{-1} x_n| \leqslant |x_n - S_n(t_0)x_n| + |S_n(t_0)x_n| + \frac{2}{h}(1 + \frac{\lambda}{h}) 2 (M + 1) h$$

$$\leq |x_n| + 2(M+1) + 4(1+\frac{\lambda}{h})(M+1) .$$

We conclude by using the fact that

$$|x_n - S_n(t)x_n| \leq 3|x_n - (I+tA_n)^{-1}x_n|.$$

<u>Proof of (35) $\Rightarrow$ (33)</u> In what follows λ is fixed. Using Theorem 8 we get

$$|(I+\frac{\lambda}{t}(I-S_n(t)))^{-1}x_n - (I+\lambda A_n)^{-1}x_n|^2$$
$$\leq |x_n - (I+\lambda A_n)^{-1}x_n| \frac{2}{t}\int_0^t |x_n - S_n(\tau)x_n|\, d\tau$$

and

$$|(I+\frac{\lambda}{t}(I-S(t)))^{-1}x - (I+\lambda A)^{-1}x|^2$$
$$\leq |x-(I+\lambda A)^{-1}x| \frac{2}{t}\int_0^t |x-S(\tau)x|\, d\tau .$$

Let $P = 2|x-(I+\lambda A)^{-1}x| + 2\sup_n |x_n - (I+\lambda A_n)^{-1}x_n| < \infty$ (by Lemma 2). We have

$$\frac{1}{t}\int_0^t |x_n - S_n(\tau)x_n|\, d\tau \leq |x_n - x| + \frac{1}{t}\int_0^t |x-S(\tau)x| + \frac{1}{t}\int_0^t |S(\tau)x - S_n(\tau)x_n|\, d\tau$$

and so

$$|(I+\lambda A_n)^{-1}x_n - (I+\lambda A)^{-1}x| \leq |(I+\frac{\lambda}{t}(I-S_n(t)))^{-1}x_n - (I+\frac{\lambda}{t}(I-S(t)))^{-1}x|$$
$$+\sqrt{P|x_n - x|} + 2\sqrt{\frac{P}{t}\int_0^t |x-S(\tau)x|\,d\tau} + \sqrt{\frac{P}{t}\int_0^t |S(\tau)x - S_n(\tau)x_n|\, d\tau}$$
$$= X_1 + X_2 + X_3 + X_4 .$$

Given $\varepsilon > 0$ we choose first $t > 0$ small enough so that $X_3 < \varepsilon$ and then we choose n large enough so that $X_1 + X_3 + X_4 < \varepsilon$ (we use here Lemma 1 to make X_1 small and Lemma 2 combined with Lebesgue's Theorem to make X_2 small).

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